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# Symmetric powers of the $p + 1$ -dimensional indecomposable module of a cyclic $p$ -group and the $\lambda$ -structure of its Green ring <sup>☆</sup>

Sun-Young Nam, Young-Tak Oh <sup>\*</sup>*Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea***ARTICLE INFO***Article history:*

Received 2 December 2011

Available online 11 July 2012

Communicated by Michel Broué

*MSC:*

20C05

20C20

*Keywords:*

Modular representation

Green ring

 $\lambda$ -Ring

Adams operation

Symmetric power

**ABSTRACT**

Let  $K$  be a field of order  $p$  and let  $G$  be a cyclic group of order  $p^k$  ( $k \geq 2$ ). We explicitly decompose the symmetric powers of the indecomposable  $KG$ -module of dimension  $p + 1$  into indecomposable  $KG$ -modules. Using this result, for every odd prime  $p$ , we give a negative answer to the conjecture posed by Kouwenhoven [F.M. Kouwenhoven, The  $\lambda$ -structure of the Green rings of cyclic  $p$ -groups, Proc. Sympos. Pure Math. 47 (1987) 451–466].

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**1. Introduction**

Let  $K$  be a field with positive characteristic  $p$  and let  $G$  be a finite group with order divisible by  $p$ . The Green ring  $RS_K(G)$  of  $G$  over  $K$  is a ring formed from isomorphism classes of the finite-dimensional  $KG$ -modules, with addition and multiplication coming from direct sums and their tensor products, respectively. If  $K$  is algebraically closed and the Sylow  $p$ -subgroups of  $G$  are cyclic, then the ring structure of  $RS_K(G)$  is theoretically approachable [6]. Even in cases of finite cyclic  $p$ -groups, however, the  $\lambda$ -structure of  $RS_K(G)$  induced from exterior powers is far from being approachable.

<sup>☆</sup> This research was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2011-0003459).

<sup>\*</sup> Corresponding author.

E-mail addresses: [synam@sogang.ac.kr](mailto:synam@sogang.ac.kr) (S.-Y. Nam), [ytok@sogang.ac.kr](mailto:ytok@sogang.ac.kr) (Y.-T. Oh).

Most obstructions to studies of the  $\lambda$ -structure of  $RS_K(G)$  are caused by the fact that  $RS_K(G)$  is not a  $\lambda$ -ring. Hence it would be quite natural to search for the smallest  $\lambda$ -ideal  $I$  such that the quotient ring  $RS_K(G)/I$  is a  $\lambda$ -ring. In 1987, Kouwenhoven settled this problem for the case in which  $G$  is a cyclic group of order  $p$  for each prime  $p$  [9]. Furthermore, he proposed a challenging conjecture for the case in which  $G$  is a cyclic group of order  $p^k$  ( $k \geq 2$ ). Throughout this paper, the Green ring is simply denoted by  $RS(p^k)$  if  $K$  is a field of order  $p$  and  $G$  is a cyclic group of order  $p^k$ . It is well known that  $RS(p^k)$  is a free  $\mathbb{Z}$ -module with basis  $\{V_n: 1 \leq n \leq p^k\}$ , where  $V_n := K[X]/(X-1)^n$ . Against this background, Kouwenhoven's conjecture is stated as follows.

**Conjecture.** (See [9].) For every prime  $p$ , let  $q = p^k$  for  $k \geq 2$  and let  $I_q$  be the ideal generated by  $V_{p^i} - V_{p^{i-1}} - V_1$  for  $1 \leq i \leq k$ . Then  $I_q$  is closed for the exterior powers and  $RS(q)/I_q$  is equipped with the  $\lambda$ -ring structure for the induced exterior powers.

It was recently shown that the above conjecture is true when  $p = 2$ , but false when  $p = 3$  [10]. We show here that it is false for every odd prime  $p$ . In brief, our strategy is to compute  $\psi_S^p(V_{p+1}) \pmod{I_q}$  in two different ways. The first is to express  $\psi_S^p(V_{p+1})$  as a linear combination of indecomposable modules using Newton's formula (for a precise description, see Theorem 4.3). It should be noted that this method is applicable since the explicit decomposition of  $S^n(V_{p+1})$  can be obtained by virtue of the elegant theorem of Shank and Wehlau [11, Theorem 1.3]. The second way is to compute  $\psi_S^p(V_{p+1})$  from  $\psi_A^p(V_{p^2-(p+1)})$  modulo  $\mathbb{Z}V_{p^2}$  using the result reported by Bryant and Johnson [5, Theorem 6.2]. Under the validity of the conjecture,  $\psi_A^p(V_{p^2-(p+1)})$  is equal to  $\psi_A^p(V_{p^2}) - \psi_A^p(V_{p+1}) \pmod{I_{p^2}}$ . Since  $\psi_A^p(V_{p^2}) = pV_p$  and  $\psi_A^p(V_{p+1})$  is computable by virtue of another result presented by Bryant and Johnson [4, Theorem 4.7], the value of  $\psi_S^p(V_{p+1}) \pmod{I_{p^2}}$  is also computable (Lemma 4.1). Finally, by comparing the results thus obtained, we derive a contradiction.

The remainder of the paper is organized as follows. Section 2 recollects the notations and theorems required to develop our arguments. Section 3 describes the explicit decomposition of  $S^n(V_{p+1})$  into a direct sum of indecomposable modules for each positive integer  $n$ . The final section shows that Kouwenhoven's conjecture does not always hold.

## 2. Preliminaries

This paper is a sequel to a previous study [10]. To avoid confusion and ambiguity, we adopt all the previous definitions and notation without change. We also introduce further notation and theorems required to develop our argument.

Let  $K$  be a field,  $G$  a finite group, and  $RS_K(G)$  the Green ring of  $G$  over  $K$ . In the case in which  $K$  is a field of order  $p$  and  $G$  is a cyclic group of order  $p^k$  for a positive integer  $k$ , we denote  $RS_K(G)$  by  $RS(p^k)$  throughout the paper. For any  $KG$ -module  $V$ , let  $V$  denote the corresponding element of  $RS_K(G)$ . It is well known that  $RS(p^k)$  is a free  $\mathbb{Z}$ -module that has a  $\mathbb{Z}$ -basis consisting of  $p^k$  indecomposable modules  $V_1, V_2, \dots, V_{p^k}$  satisfying  $\dim V_n = n$  for each  $1 \leq n \leq p^k$ . Conventionally,  $V_0$  is defined by the zero element in  $RS(p^k)$ .

The most significant pre- $\lambda$ -ring structures of  $RS_K(G)$  come from the exterior powers  $\Lambda^n$  and the symmetric powers  $S^n$  ( $n = 0, 1, 2, \dots$ ), respectively. For each positive integer  $n$  and for all  $x \in RS_K(G)$ , we define  $\psi_A^n(x)$  and  $\psi_S^n(x)$  as follows.

$$\sum_{n=0}^{\infty} (-1)^n \psi_A^{n+1}(x) t^n := \frac{d}{dt} \log \Lambda_t(x),$$

$$\sum_{n=0}^{\infty} \psi_S^{n+1}(x) t^n := \frac{d}{dt} \log S_t(x),$$

where

$$\begin{aligned}\Lambda_t(x) &= 1 + \Lambda^1(x)t + \Lambda^2(x)t^2 + \cdots, \\ S_t(x) &= 1 + S^1(x)t + S^2(x)t^2 + \cdots.\end{aligned}$$

We can easily show that  $\psi_A^n(x), \psi_S^n(x) \in RS_K(G)$  and

$$\psi_A^n(x+y) = \psi_A^n(x) + \psi_A^n(y), \quad \psi_S^n(x+y) = \psi_S^n(x) + \psi_S^n(y)$$

for all positive integers  $n$  and all  $x, y \in RS_K(G)$ . Hence, we have the  $\mathbb{Z}$ -linear functions

$$\psi_A^n, \psi_S^n : RS_K(G) \rightarrow RS_K(G),$$

which are called the  $n$ th Adams operations on  $RS_K(G)$ . From the definition of Adams operations, we can derive Newton's formula for  $\psi_A^n$  and  $\psi_S^n$  ( $n \geq 1$ ):

$$\sum_{i=0}^{n-1} (-1)^{i+1} \psi_A^{n-i} \Lambda^i = (-1)^n n \Lambda^n \quad \text{and} \quad \sum_{i=0}^{n-1} \psi_S^{n-i} S^i = n S^n, \quad \forall n \geq 1. \quad (2.1)$$

These formulae imply that each  $\psi_A^n$  and  $\psi_S^n$  can be expressed as a polynomial in  $\Lambda^i$  and  $S^i$  ( $1 \leq i \leq n$ ), respectively, with integer coefficients. As a consequence, if  $I$  is invariant under  $\Lambda^n$  ( $S^n$ ) for all  $n \geq 1$ , then it is also invariant under  $\psi_A^n$  ( $\psi_S^n$ ) for all  $n \geq 1$ . In studies of the  $\lambda$ -structure of  $RS_K(G)$ , Adams operations are useful because they contain much information about  $\lambda$ -operations and are easier to deal with than  $\lambda$ -operations because of their additivity. In particular, Bryant showed that  $\psi_A^n$  and  $\psi_S^n$  behave well provided  $n$  is not divisible by the characteristic of  $K$  [3].

**Theorem 2.1.** (See [3].) *For every positive integer  $n$  that is not divisible by the characteristic of a field  $K$ , we have  $\psi_A^n = \psi_S^n$  and each of these maps is a ring endomorphism of  $RS_K(G)$ . Furthermore, under the composition of maps we have*

$$\psi_A^n \circ \psi_A^m = \psi_A^{nm}, \quad \psi_S^n \circ \psi_S^m = \psi_S^{nm}$$

for all positive integers  $m$ .

For a  $KG$ -module  $V$ , let  $\mathcal{P}_V$  denote the projective cover of  $V$ . It is uniquely determined up to isomorphism and the value of the Heller operation  $\Omega$  at  $V$  is defined as the kernel of the map  $\mathcal{P}_V \rightarrow V$  so that the short sequence

$$0 \rightarrow \Omega(V) \rightarrow \mathcal{P}_V \rightarrow V \rightarrow 0$$

is exact [2]. Then we can have a  $\mathbb{Z}$ -linear function on  $RS_K(G)$  by extending  $\Omega$  by linearity. We define  $\Omega^0$  as the identity map and  $\Omega^n$  the map obtained by composing  $\Omega$   $n$  times for each positive integer  $n$ . In the case in which  $K = \mathbb{F}_p$  and  $G$  is a cyclic group of order  $p^k$ ,  $V_{p^k}$  is the only projective indecomposable  $KG$ -module and  $V_{p^k-r}$  is the Heller translate of  $V_r$  in  $RS(p^k)$ .

For any positive integer  $n$  not divisible by  $p$ , let  $\gamma(n)$  denote the unique integer satisfying the conditions  $1 \leq \gamma(n) \leq p-1$  and  $n \equiv \gamma(n)$  (or,  $-\gamma(n)$ ) (mod  $2p$ ) and set  $\gamma(0)$  to 0. For  $m \in \{0, \dots, k-1\}$  and  $i \in \{1, \dots, p-1\}$ , we define a  $\mathbb{Z}$ -linear map  $\theta_{ip^m} : RS(p^m) \rightarrow RS(p^{m+1})$  so that

$$\theta_{ip^m}(V_r) = V_{ip^{m+r}} - V_{ip^{m-r}}$$

for  $r = 1, \dots, p^m$ . In particular,  $\theta_0$  is defined as the identity map on  $RS(p^m)$ . The following propositions are useful for computation of the value of Adams operations.

**Proposition 2.2.** (See [4,5].) *Let  $k$  be a positive integer. Then we have the following.*

- (a) *Let  $m \in \{0, \dots, k-1\}$  and let  $n$  be a positive integer not divisible by  $p$ . Let  $s$  be a positive integer satisfying  $p^m < s \leq p^{m+1}$  and write  $s = s_0 p^m + s_1$ , where  $1 \leq s_0 \leq p-1$  and  $1 \leq s_1 \leq p^m$ . Then*

$$\psi_A^n(V_s) = \sum_{\substack{i \in \{0, \dots, s_0\} \\ i \equiv s_0 \pmod{2}}} \theta_{\gamma(in)p^m}(\psi_A^n(V_{s_1})) + \sum_{\substack{i \in \{0, \dots, s_0\} \\ i \not\equiv s_0 \pmod{2}}} \theta_{\gamma(in)p^m}(\psi_A^n(V_{p^m-s_1})).$$

- (b) *Let  $q = p^k$ . Then for each positive integer  $n$ , the  $n$ th Adams operations  $\psi_A^n$  and  $\psi_S^n$  on  $RS(q)$  satisfy the relation*

$$\psi_S^n(V_s) \equiv (-1)^{n-1} \Omega^n(\psi_A^n(V_{q-s})) + (n, q) V_{n/(n,q)} \pmod{\mathbb{Z}V_q},$$

where  $q/p \leq s \leq q$ .

**Proposition 2.3.** (See [4,5].) *Let  $k$  be a positive integer. Then we have the following.*

- (a) *For  $0 \leq i \leq k-1$ ,  $\chi_i \psi_A^p(\chi_i) = 2\psi_A^{p-1}(\chi_i)$  holds, where  $\chi_i$  denotes the  $i$ th generator of  $RS(p^k)$ , i.e.,  $V_{p^{i+1}} - V_{p^{i-1}}$ .*  
 (b) *Let  $q$  be a power of an odd prime. Then for each positive integer  $n$ ,  $\psi_A^n(V_q) = (n, q) V_{q/(n,q)}$  holds.*  
 (c) *For all positive integers  $n$  not divisible by  $p$  and  $1 \leq m \leq k$ ,*

$$\psi_A^n(V_{p^m-1}) = \begin{cases} V_{p^m-1} & \text{if } n \text{ is odd,} \\ V_{p^m} - V_1 & \text{if } n \text{ is even.} \end{cases}$$

The following proposition presents the rules necessary for multiplication of  $RS(q)$  through decomposition of the tensor product of two indecomposable modules.

**Proposition 2.4.** (See [8,9].) *Suppose that  $p$  is prime and  $q = p^k$ , where  $k$  is a positive integer.*

- (a) *For  $0 \leq n \leq q$ , we have*

$$V_2 V_n = \begin{cases} V_{n-1} + V_{n+1} & \text{if } p \nmid n, \\ 2V_n & \text{if } p \mid n. \end{cases}$$

- (b) *Let  $m$  and  $n$  be positive integers such that  $m \leq n \leq pq$ . We write  $n = n_0 q + n_1$  and  $m = m_0 q + m_1$  with  $0 \leq m_1, n_1 \leq q-1$ . Suppose  $V_{n_1} V_{m_1} = \sum_s a_s V_s$ . If  $m+n \leq pq$ , then*

$$\begin{aligned} V_n V_m &= \sum_{i=1}^{m_0} \sum_{s=1}^{q-1} a_s (V_{(n_0+m_0+2-2i)q+s} + V_{(n_0+m_0+2-2i)q-s}) \\ &\quad + \sum_{s=1}^{q-1} a_s V_{(n_0-m_0)q+s} + |n_1 + m_1 - q| \sum_{i=1}^{m_0} V_{(n_0+m_0+1-2i)q} \\ &\quad + |m_1 - n_1| \sum_{i=1}^{m_0} V_{(n_0+m_0+2-2i)q} + \max(0, m_1 + n_1 - q) V_{(n_0+m_0+1)q} \\ &\quad + \max(0, m_1 - n_1) V_{(n_0-m_0)q}. \end{aligned}$$

If  $m + n > pq$ , then

$$V_n V_m = V_{pq-n} V_{pq-m} + (n + m - pq) V_{pq}.$$

It has been shown that there exists a ring isomorphism, say  $\varphi : RS(p)^{\otimes k} \rightarrow RS(q)$ , defined by [1,7]

$$1 \otimes \cdots \otimes 1 \otimes \overbrace{V_2}^{\text{ith}} \otimes 1 \otimes \cdots \otimes 1 \mapsto V_{2p^{i-1}} - V_{2p^{i-1}-1} + V_1 \quad (1 \leq i \leq k).$$

Contrary to the simplicity of the ring structure, the  $\lambda$ -structure of  $RS(q)$  is extremely complicated. Indeed  $RS(p^k)$  ( $k \geq 1$ ) is not a  $\lambda$ -ring for the  $\lambda$ -operations coming from the exterior powers. We define  $I_p$  as the ideal of  $RS(p)$  generated by  $V_p - V_{p-1} - V_1$ . In 1987, Kouwenhoven showed that  $RS(p)/I_p$  is the largest  $\mathbb{Z}$ -torsion free quotient of  $RS(p)$  that is a  $\lambda$ -ring for the induced exterior powers [9]. He also showed that  $\varphi$  induces a ring isomorphism

$$\bar{\varphi} : (RS(p)/I_p)^{\otimes k} \rightarrow RS(q)/I_q, \quad \bar{x}_1 \otimes \bar{x}_2 \otimes \cdots \otimes \bar{x}_k \mapsto \overline{\varphi(x_1 \otimes x_2 \otimes \cdots \otimes x_k)},$$

where  $I_q$  is the ideal generated by  $V_{p^i} - V_{p^i-1} - V_1$  for all  $i$  with  $1 \leq i \leq k$ . Here, bar notation is used to denote a coset. However, if there is no danger of confusion, this will be omitted for simplicity. Motivated by this observation, he proposed the following conjecture on the  $\lambda$ -structure of  $RS(q)/I_q$  when  $k \geq 2$ .

**Conjecture 2.5.** (See [9].) For every prime  $p$ , let  $q = p^k$  with  $k \geq 2$  and let  $I_q$  be the ideal generated by  $V_{p^i} - V_{p^i-1} - V_1$  for  $1 \leq i \leq k$ . The exterior powers induce operations on  $RS(q)/I_q$  and the induced ring isomorphism  $\bar{\varphi} : (RS(p)/I_p)^{\otimes k} \xrightarrow{\cong} RS(q)/I_q$  commutes with the  $\lambda$ -operations. In particular,  $RS(q)/I_q$  is a  $\lambda$ -ring.

We recently presented a partial result for Conjecture 2.5 [10]: we showed that it is true when  $p = 2$ , but false when  $p = 3$ . The main goal of the present paper is to show that  $I_{p^k}$  is not closed for the exterior powers if  $p$  is an odd prime and  $k \geq 2$ .

We finish this section by providing a  $\mathbb{Z}$ -basis of  $RS(q)/I_q$ , which plays a key role in the final step in disproving Conjecture 2.5.

Let

$$\mathcal{A} := \left\{ (a_1, \dots, a_{k-1}) : 0 \leq a_i \leq \frac{p-1}{2} \text{ for all } 1 \leq i \leq k-1 \right\}.$$

For any  $(k-1)$ -tuple  $(a_1, \dots, a_{k-1}) \in \mathcal{A}$ , we define  $\mathbf{sum}(a_1, \dots, a_{k-1})$  as

$$\sum_{1 \leq m \leq k-1} a_m p^m.$$

**Proposition 2.6.** (See [10].) Let  $k$  be a positive integer and let  $q = p^k$ . Then the set

$$\{V_{\mathbf{sum}(a_1, \dots, a_{k-1})+j} : (a_1, \dots, a_{k-1}) \in \mathcal{A}, j = 1, 3, \dots, p\}$$

is a  $\mathbb{Z}$ -basis of  $RS(q)/I_q$ .

### 3. Symmetric powers of $V_{p+1}$

In this section, we decompose  $S^n(V_{p+1})$  into a direct sum of indecomposable modules for each positive integer  $n$ . It should be noted that Shank and Wehlau have already succeeded in decomposing  $S^n(V_{p+1})$  into a direct sum of indecomposable modules (Theorem 3.4) [11]. Their decomposition, however, is up to induced modules. Here we calculate the multiplicity of each induced module appearing in the decomposition of  $S^n(V_{p+1})$  precisely.

We first provide two results on the decomposition of symmetric powers.

**Proposition 3.1.** (See [1].)

- (a)  $S^r(V_2) \cong V_{r+1}$  for  $0 \leq r \leq p-1$ .
- (b)  $S^r(V_{p^m-t}) \cong S^{r/p}(V_{p^{m-1}}) \oplus \text{free}$  for  $r \equiv 0 \pmod{p}$  and  $t$  in the range  $0 \leq t \leq p-1$ .

**Lemma 3.2.** For any non-negative integer  $s$  and  $0 \leq r \leq p-1$ , we have

$$S^{sp+r}(V_2) \cong V_{r+1} \oplus sV_p. \quad (3.1)$$

**Proof.** If  $r = 0$ , then  $S^{sp}(V_2) \cong S^s(V_1) \oplus \text{free}$ , which follows from Proposition 3.1(b), and this holds for all prime  $p$ . Comparing the dimension of either side of (3.1) immediately yields the desired result. Thus,  $r$  is assumed to be positive hereafter. To accomplish our purpose, we use mathematical induction on  $sp+r$ . Note that  $S^1(V_2) \cong V_2$ . Suppose that the decomposition of  $S^k(V_2)$  satisfies our assertion for all  $k < sp+r$ , where  $sp+r > 1$ . Since  $sp+r$  is not divisible by  $p$ , in this case we have

$$\sum_{i=0}^{sp+r} (-1)^i \Delta^i(V_2) S^{sp+r-i}(V_2) = 0 \quad (3.2)$$

in  $RS(p)$ . Thus,

$$S^{sp+r}(V_2) - V_2 S^{sp+r-1}(V_2) + S^{sp+r-2}(V_2) = 0.$$

If  $p \neq 2$ , the induction hypothesis implies that

$$\begin{aligned} 0 &= S^{sp+r}(V_2) - V_2(V_r + sV_p) + S^{sp+r-2}(V_2) \\ &= \begin{cases} S^{sp+r}(V_2) - V_2 - 2sV_p + V_p + (s-1)V_p & \text{if } r = 1, \\ S^{sp+r}(V_2) - V_{r+1} - V_{r-1} - 2sV_p + V_{r-1} + sV_p & \text{otherwise} \end{cases} \\ &= \begin{cases} S^{sp+r}(V_2) - V_2 - sV_p & \text{if } r = 1, \\ S^{sp+r}(V_2) - V_{r+1} - sV_p & \text{otherwise.} \end{cases} \end{aligned}$$

The second equality follows from Proposition 2.4(a). Conversely, if  $p = 2$ , we have

$$\begin{aligned} 0 &= S^{sp+1}(V_2) - V_2(V_1 + sV_2) + S^{(s-1)p+1}(V_2) \\ &= S^{sp+1}(V_2) - V_2 - 2sV_2 + sV_2 \\ &= S^{sp+1}(V_2) - (s+1)V_2. \end{aligned}$$

This completes the proof.  $\square$

**Remark 3.3.** For any  $KG$ -module  $V$ , consider the following exact sequence:

$$0 \rightarrow \Lambda^d(V) \rightarrow \Lambda^{d-1} \otimes S^1(V) \rightarrow \cdots \rightarrow \Lambda^1(V) \otimes S^{d-1}(V) \rightarrow S^d(V) \rightarrow 0.$$

It is well known that the above exact sequence splits whenever  $d$  is invertible, and hence Eq. (3.2) follows [1, Theorem 2.3].

Next we introduce a result due to Shank and Wehlau on the decomposition of  $S^n(V_{p+1})$  into a direct sum of indecomposable modules [11]. We first review the notation required. Let  $G$  be a finite group and let  $H$  be a subgroup of  $G$ . For any  $KG$ -module  $V$ , let  $\text{Res}_H(V)$  denote the  $KH$ -module obtained from  $V$  by restriction. Since restriction commutes with direct sums and tensor products, we can extend  $\text{Res}_H$  to a ring homomorphism from  $RS_K(G)$  to  $RS_K(H)$ . Furthermore, it is well known that  $\text{Res}_H$  commutes with the exterior powers and symmetric powers. Conversely, for any  $KH$ -module  $W$  let  $\text{Ind}^G(W)$  denote the  $KG$ -module obtained from  $W$  by induction. Since induction commutes with direct sums,  $\text{Ind}^G$  can also be extended to a  $\mathbb{Z}$ -linear map from  $RS_K(H)$  to  $RS_K(G)$ . A  $KG$ -module is said to be induced if it is induced from a  $KH$ -module for some subgroup  $H$  of  $G$  and an element of  $RS_K(G)$  is said to be induced if it is a  $\mathbb{Z}$ -linear combination of induced modules. Now assume that  $K$  is a field of characteristic  $p$ ,  $G$  is a cyclic group of order  $p^k$ , and  $H$  is the unique subgroup of index  $p$  in  $G$ . Then we have

$$\text{Res}_H(V_s) = s_1 V_{s_0+1} + (p - s_1) V_{s_0}, \quad (3.3)$$

where  $s = s_0 p + s_1$  with  $0 \leq s_1 \leq p - 1$  and, for  $r = 1, \dots, p^{k-1}$ ,

$$\text{Ind}^G(V_r) = V_{rp}.$$

It is not difficult to show that  $V_r$  is induced from a module of a proper subgroup of  $G$  if and only if  $r$  is divisible by  $p$ . Hereafter, we write  $\text{Res}$  for  $\text{Res}_H$ , the restriction map (3.3), for simplicity. The following theorem is key in the proof of our main result.

**Theorem 3.4.** (See [11].) Let  $K$  be a field of characteristic  $p$ , let  $G$  be a cyclic group of order  $p^2$  and let  $d$  be any non-negative integer. In the decomposition of  $S^n(V_{p+1})$  into a direct sum of indecomposable  $KG$ -modules, there is at most one indecomposable summand  $V_r$  that is not induced from a representation of a proper subgroup. In particular, writing  $n = ap^2 + bp + c$ , where  $0 \leq b, c \leq p - 1$ ,  $S^n(V_{p+1})$  is an induced module when  $b = p - 1$  and there is exactly one non-induced indecomposable summand when  $b \leq p - 2$  that is isomorphic to  $V_{cp+b+1}$ .

Note that

$$\begin{aligned} S_t(\text{Res}(V_{p+1})) &= S_t((p-1)V_1 + V_2) \\ &= (1 + t + t^2 + \cdots)^{p-1} (1 + S^1(V_2)t + S^2(V_2)t^2 + \cdots). \end{aligned}$$

Thus, the coefficient of  $t^n$  in  $S_t(\text{Res}(V_{p+1}))$  is given by

$$S^n(V_2) + \binom{p-1}{1} S^{n-1}(V_2) + \cdots + \binom{p-1}{n-1} S^1(V_2) + \binom{p-1}{n} V_1, \quad (3.4)$$

where the notation  $\binom{n}{k}$  denotes the number of  $k$ -multicombinations (or  $k$ -combinations with repetitions) of an  $n$ -element set. Note that  $\binom{n}{k} = \binom{n+k-1}{k}$ . To apply Lemma 3.2, we write the above summation as

$$\begin{aligned} & \sum_{j=0}^{c+1} \left( \binom{p-1}{j} \right) S^{n-j}(V_2) + \sum_{i=0}^{ap+b-2} \sum_{j=0}^{p-1} \left( \binom{p-1}{c+2+pi+j} \right) S^{n-(c+2+pi+j)}(V_2) \\ & + \sum_{j=0}^{p-2} \left( \binom{p-1}{n+2-p+j} \right) S^{n-(n+2-p+j)}(V_2), \end{aligned} \quad (3.5)$$

where  $n = ap^2 + bp + c$  with  $0 \leq b, c \leq p-1$ . Using the convention that  $\binom{n}{k}$  is zero when  $k$  is a negative integer, in view of Lemma 3.2, we can write the summation in Eq. (3.5) as a linear combination

$$\sum_{k=1}^p c(k) V_k$$

of  $V_1, \dots, V_p$ , where

$$c(k) = \sum_{i=0}^{ap+b} \left( \binom{p-1}{c+1-k+ip} \right), \quad 1 \leq k \leq p-1$$

and

$$c(p) = \begin{cases} \sum_{i=0}^{ap+b} (ap+b-i) \sum_{j=0}^{p-1} \left( \binom{p-1}{c+1-j+ip} \right) & \text{if } c \neq p-1, \\ \sum_{i=0}^{ap+b} (ap+b-i) \sum_{j=0}^{p-1} \left( \binom{p-1}{c+1-j+ip} \right) + ap+b+1 & \text{if } c = p-1. \end{cases} \quad (3.6)$$

Moreover, for  $0 \leq c \leq p-1$  and  $0 \leq j \leq p-1$ , it is easy to show that  $\binom{p-1}{c+1-j-p} = 0$  unless  $c = p-1$  and  $j = 0$ . Thus, two cases in Eq. (3.6) can be merged as

$$c(p) = \sum_{i=-1}^{ap+b} (ap+b-i) \sum_{j=0}^{p-1} \left( \binom{p-1}{c+1-j+ip} \right).$$

Next, we recall the identity

$$S^n(\text{Res}(V_{p+1})) = \text{Res}(S^n(V_{p+1})),$$

which follows from the fact that the restriction map commutes with symmetric power. Using Theorem 3.4, we can derive

$$S^n(V_{p+1}) = \begin{cases} V_{cp+b+1} + c_n(1)V_p + c_n(2)V_{2p} + \dots + c_n(p)V_{p^2} & \text{if } b \leq p-2, \\ c_n(1)V_p + c_n(2)V_{2p} + \dots + c_n(p)V_{p^2} & \text{if } b = p-1, \end{cases} \quad (3.7)$$

for some non-negative integers  $c_n(i)$  ( $1 \leq i \leq p$ ). To Eq. (3.7) we apply the formulae

$$\text{Res}(V_{cp+b+1}) = (p-b-1)V_c + (b+1)V_{c+1}$$

and

$$\text{Res}(V_{kp}) = pV_k$$



and then compare the coefficient of  $V_i$  on either side of  $S^n(\text{Res}(V_{p+1})) = \text{Res}(S^n(V_{p+1}))$ . This enables us to derive the explicit decomposition of  $S^n(V_{p+1})$  into indecomposables.

**Theorem 3.5.** Writing  $n = ap^2 + bp + c$  with  $0 \leq b, c \leq p - 1$ , we have

$$S^n(V_{p+1}) = (1 - \delta_{b,p-1})V_{cp+b+1} + \sum_{k=1}^p c_n(k)V_{kp},$$

where if  $b = p - 1$ , then

$$pc_n(k) = \begin{cases} \sum_{i=0}^{ap+b} \binom{p-1}{c+1-k+ip} & \text{if } 1 \leq k \leq p-1, \\ \sum_{i=-1}^{ap+b} (ap+b-i) \sum_{j=0}^{p-1} \binom{p-1}{c+1-j+ip} & \text{if } k = p \end{cases}$$

and if  $b \neq p - 1$ , then

$$pc_n(k) = \begin{cases} \sum_{i=0}^{ap+b} \binom{p-1}{c+1-k+ip} & \text{if } 1 \leq k \leq p-1, k \neq c+1, \\ \sum_{i=0}^{ap+b} \binom{p-1}{1+ip} - p + b + 1 & \text{if } k = c, \\ \sum_{i=0}^{ap+b} \binom{p-1}{ip} - b - 1 & \text{if } k = c+1, k \neq p, \\ \sum_{i=-1}^{ap+b} (ap+b-i) \sum_{j=0}^{p-1} \binom{p-1}{(i+1)p-j} - b - 1 & \text{if } k = c+1, k = p, \\ \sum_{i=-1}^{ap+b} (ap+b-i) \sum_{j=0}^{p-1} \binom{p-1}{c+1-j+ip} & \text{if } k \neq c+1, k = p. \end{cases}$$

**Example 3.6.** Let  $p = 5$ . According to Theorem 3.5, the first ten  $S^n(V_6)$  value ( $1 \leq n \leq 10$ ) are decomposed into indecomposable modules in the following fashion.

$S^1(V_6)$	$V_6$
$S^2(V_6)$	$V_{11} \oplus 2V_5$
$S^3(V_6)$	$V_{16} \oplus 4V_5$
$S^4(V_6)$	$V_{21} \oplus 7V_5 \oplus 4V_{10} \oplus 2V_{15}$
$S^5(V_6)$	$V_2 \oplus 11V_5 \oplus 7V_{10} \oplus 4V_{15} \oplus 2V_{20} \oplus V_{25}$
$S^6(V_6)$	$V_7 \oplus 17V_5 \oplus 11V_{10} \oplus 7V_{15} \oplus 4V_{20} \oplus 3V_{25}$
$S^7(V_6)$	$V_{12} \oplus 26V_5 \oplus 17V_{10} \oplus 11V_{15} \oplus 7V_{20} \oplus 7V_{25}$
$S^8(V_6)$	$V_{17} \oplus 37V_5 \oplus 26V_{10} \oplus 17V_{15} \oplus 11V_{20} \oplus 14V_{25}$
$S^9(V_6)$	$V_{22} \oplus 51V_5 \oplus 37V_{10} \oplus 26V_{15} \oplus 17V_{20} \oplus 25V_{25}$
$S^{10}(V_6)$	$V_3 \oplus 68V_5 \oplus 51V_{10} \oplus 37V_{15} \oplus 26V_{20} \oplus 43V_{25}$

#### 4. The value of Adams operations at $V_{p+1}$ and Kouwenhoven's conjecture

The purpose of this section is to disprove Conjecture 2.5 for every odd prime. We first provide a brief outline of our argument. Assume that  $I_q$  is closed for the exterior powers. Then it is also closed for all Adams operations associated with the exterior powers, that is,  $\psi_\Lambda^n(I_q) \subseteq I_q$  for all positive integers  $n$ , because each  $\psi_\Lambda^n$  can be expressed as a polynomial in  $\lambda^n$  ( $n \geq 1$ ) with integer coefficients. Consequently,  $\psi_\Lambda^p(V_{p^2-(p+1)})$  should be equal to  $\psi_\Lambda^p(V_{p^2} - V_{p+1})$  modulo  $I_{p^2}$  since  $V_{p+1}(V_{p^2} - V_{p^2-1} - V_1) = V_{p^2} - V_{p^2-(p+1)} - V_{p+1} \in I_{p^2}$ . However, we show that this phenomenon does not occur if  $p$  is odd. Throughout this section,  $p$  denotes an odd prime.

First, we express  $\psi_\Lambda^p(V_{p^2} - V_{p+1})$  as a linear combination of  $V_i$  values modulo  $I_{p^2}$ . To use Proposition 2.3(a), we multiply by  $V_{p+1} - V_{p-1}$ . Since  $\psi_\Lambda^p(V_{p^2}) = pV_p$  by Proposition 2.3(b), it follows that

$$\begin{aligned}
& (V_{p+1} - V_{p-1})\psi_A^p(V_{p^2} - V_{p+1}) \\
&= (V_{p+1} - V_{p-1})(pV_p - \psi_A^p(V_{p+1})) \\
&= pV_p(V_{p+1} - V_{p-1}) - (V_{p+1} - V_{p-1})\psi_A^p(V_{p+1} - V_{p-1}) \\
&\quad - (V_{p+1} - V_{p-1})\psi_A^p(V_{p-1}).
\end{aligned} \tag{4.1}$$

The term  $(V_{p+1} - V_{p-1})\psi_A^p(V_{p+1} - V_{p-1})$  is equal to  $2\psi_A^{p-1}(V_{p+1} - V_{p-1})$  by Proposition 2.3(a) because  $\chi_1 = V_{p+1} - V_{p-1}$ . Furthermore, from [3, Lemma 3.4] it follows that  $\psi_A^p(V_{p-1}) \equiv (p-1)V_1 \pmod{I_{p^2}}$ . Substituting these into Eq. (4.1) yields the modulo equivalence

$$\begin{aligned}
& (V_{p+1} - V_{p-1})\psi_A^p(V_{p^2} - V_{p+1}) \\
&\equiv pV_p(V_{p+1} - V_{p-1}) - 2\psi_A^{p-1}(V_{p+1} - V_{p-1}) \\
&\quad - (p-1)(V_{p+1} - V_{p-1}) \pmod{I_{p^2}}.
\end{aligned} \tag{4.2}$$

Moreover, by Propositions 2.3(c) and 2.2(b), we can deduce that  $\psi_A^{p-1}(V_{p-1}) = V_p - V_1$ , and hence

$$\begin{aligned}
\psi_A^n(V_{p+1}) &= \theta_{\gamma(n)p}(\psi^n(V_1)) + \theta_{\gamma(0)p}(\psi_A^n(V_{p-1})) \\
&= V_{np+1} - V_{np-1} + \psi_A^n(V_{p-1}) \\
&= \begin{cases} V_{np+1} - V_{np-1} + V_{p-1} & \text{if } n \text{ is odd,} \\ V_{np+1} - V_{np-1} + V_p - V_1 & \text{if } n \text{ is even,} \end{cases}
\end{aligned} \tag{4.3}$$

where  $1 \leq n \leq p-1$ . Applying Eq. (4.3) to Eq. (4.2), we finally have the following identities.

**Lemma 4.1.**

$$(V_{p+1} - V_{p-1})\psi_A^p(V_{p^2} - V_{p+1}) \equiv pV_p(V_{p+1} - V_{p-1}) - (p+1)(V_{p+1} - V_{p-1}) \pmod{I_{p^2}}$$

and

$$\psi_A^{p-1}(V_{p+1}) \equiv V_{p+1} \pmod{I_{p^2}}.$$

Letting  $q = p^2$ ,  $s = p^2 - (p+1)$  in Proposition 2.2(b) yields the identity

$$(V_{p+1} - V_{p-1})\psi_A^p(V_{p^2-(p+1)}) \equiv (V_{p+1} - V_{p-1})(\Omega(\psi_S^p(V_{p+1}) - pV_p)) \pmod{\mathbb{Z}V_{p^2}}. \tag{4.4}$$

Note that  $\Omega(V_r) \equiv V_{p^2} - V_r \pmod{I_{p^2}}$  since the Heller operation  $\Omega$  translates  $V_r$  to  $V_{p^2-r}$  in  $RS(p^2)$ . Therefore, we have the following lemma.

**Lemma 4.2.**

$$\begin{aligned}
& (V_{p+1} - V_{p-1})\psi_A^p(V_{p^2-(p+1)}) \\
&\equiv pV_p(V_{p+1} - V_{p-1}) - \psi_S^p(V_{p+1})(V_{p+1} - V_{p-1}) \pmod{I_{p^2}}.
\end{aligned} \tag{4.5}$$

**Proof.** The desired result can be obtained by comparing the dimension on either side of Eq. (4.5).  $\square$

Thus, if Conjecture 2.5 is true, then by Lemmas 4.1 and 4.2,

$$\psi_S^p(V_{p+1})(V_{p+1} - V_{p-1}) \equiv (p+1)(V_{p+1} - V_{p-1}) \pmod{I_{p^2}}.$$

We now show that this modulo equivalence does not hold unless  $p = 2$  by computing the explicit value of  $\psi_S^p$  at  $V_{p+1}$ .

We first introduce multiplication formulae necessary for computation of  $\psi_S^p(V_{p+1})$ , all of which can be derived from Proposition 2.4 by direct calculation.

(P1) If  $s > r$ , then

$$\begin{aligned} (V_{sp+1} - V_{sp-1} + V_{p-1})V_{rp} &= (V_{sp+1} - V_{sp-1} + V_p - V_1)V_{rp} \\ &= V_{(s+r)p} - V_{(s-r)p} + (p-1)V_{rp}. \end{aligned}$$

(P2) If  $s = r$ , then

$$\begin{aligned} (V_{sp+1} - V_{sp-1} + V_{p-1})V_{rp} &= (V_{sp+1} - V_{sp-1} + V_p - V_1)V_{rp} \\ &= V_{2rp} + (p-1)V_{rp}. \end{aligned}$$

(P3) If  $s < r$ , then

$$\begin{aligned} (V_{sp+1} - V_{sp-1} + V_{p-1})V_{rp} &= (V_{sp+1} - V_{sp-1} + V_p - V_1)V_{rp} \\ &= V_{(s+r)p} + V_{(r-s)p} + (p-1)V_{rp}. \end{aligned}$$

(P4) If  $1 \leq r \leq \frac{p-1}{2}$ , then

$$\begin{aligned} (V_{(p-r)p+1} - V_{(p-r)p-1} + V_{p-1})V_{rp+1} \\ = (p-2)V_{rp} - V_{p^2-1} + 2V_{p^2} + V_{(r+1)p-1} - V_{(p-2r)p-1}. \end{aligned}$$

(P5) If  $1 \leq r \leq \frac{p-1}{2}$ , then

$$\begin{aligned} (V_{(p-r)p+1} - V_{(p-r)p-1} + V_p - V_1)V_{rp+1} \\ = (p-1)V_{rp} - V_{p^2-1} + 2V_{p^2} + V_{(r+1)p} - V_{(p-2r)p-1} - V_{rp+1}. \end{aligned}$$

(P6) If  $\frac{p+1}{2} \leq r \leq p-1$ , then

$$\begin{aligned} (V_{(p-r)p+1} - V_{(p-r)p-1} + V_{p-1})V_{rp+1} \\ = (p-2)V_{rp} - V_{p^2-1} + 2V_{p^2} + V_{(r+1)p-1} + V_{(2r-p)p+1}. \end{aligned}$$

(P7) If  $\frac{p+1}{2} \leq r \leq p-1$ , then

$$\begin{aligned} (V_{(p-r)p+1} - V_{(p-r)p-1} + V_p - V_1)V_{rp+1} \\ = (p-1)V_{rp} - V_{p^2-1} + 2V_{p^2} + V_{(r+1)p} + V_{(2r-p)p+1} - V_{rp+1}. \end{aligned}$$

**Theorem 4.3.** For every odd prime  $p$ , we have

$$\psi_S^p(V_{p+1}) = pV_2 + V_{p-1} - V_p + (p-2)V_{p^2-1} - (p-2)V_{p^2}.$$

**Proof.** In view of Theorem 3.5, we obtain

$$S^i(V_{p+1}) = V_{ip+1} + \sum_{r=1}^{i-1} \binom{p-1+i-r}{p-2} p^{-1} V_{rp}, \quad (4.6)$$

where  $1 \leq i \leq p-1$ , and

$$S^p(V_{p+1}) = V_2 + \left( \binom{2p-2}{p-2} - 1 \right) p^{-1} V_p + \sum_{r=2}^{p-1} \binom{2p-1-r}{p-2} p^{-1} V_{rp} + V_{p^2}. \quad (4.7)$$

In addition, Eq. (2.1) implies that

$$\psi_S^p(V_{p+1}) = pS^p(V_{p+1}) - \sum_{i=1}^{p-1} \psi_S^{p-i}(V_{p+1}) S^i(V_{p+1}).$$

For any  $V$  in  $RS(p^k)$ , let  $[V]_{\text{Ind}}$  denote the sum of induced indecomposable summands in the decomposition of  $V$  into indecomposables. More precisely, if  $V = \sum_{i \geq 1} c_i V_i$ , then  $[V]_{\text{Ind}}$  is defined as  $\sum_{i \geq 1} c_{pi} V_{pi}$ . Using this notation, we can rewrite  $\psi_S^p(V_{p+1})$  as

$$pV_2 + p[S^p(V_{p+1})]_{\text{Ind}} - \sum_{i=1}^{p-1} \psi_S^{p-i}(V_{p+1}) V_{ip+1} - \sum_{i=1}^{p-1} \psi_S^{p-i}(V_{p+1}) [S^i(V_{p+1})]_{\text{Ind}}. \quad (4.8)$$

To express Eq. (4.8) as a linear combination of indecomposables, we first focus on the last term,

$$\sum_{i=1}^{p-1} \psi_S^{p-i}(V_{p+1}) [S^i(V_{p+1})]_{\text{Ind}}.$$

Utilizing Theorem 2.1 and Eq. (4.3), we can show that  $\psi_S^{p-i}(V_{p+1})$  is equal to

$$\begin{cases} V_{sp+1} - V_{sp-1} + V_{p-1} & \text{if } s \text{ is odd,} \\ V_{sp+1} - V_{sp-1} + V_p - V_1 & \text{if } s \text{ is even,} \end{cases}$$

where  $s = p - i$ . We then multiply  $\psi_S^{p-i}(V_{p+1})$  by  $[S^i(V_{p+1})]_{\text{Ind}}$  using formulae (P1), (P2) and (P3). For each  $1 \leq i \leq p-1$ , the multiplicity of  $V_{ep}$  ( $1 \leq e \leq i-1$ ) in  $S^i(V_{p+1})$  is given by

$$m(i, e) := \binom{p-1+i-e}{p-2} p^{-1}$$

in view of Eq. (4.6). For  $e \leq i-1$ , we set  $m(i, e)$  to zero. Therefore, we obtain

$$\psi_S^{p-i}(V_{p+1}) [S^i(V_{p+1})]_{\text{Ind}} = \sum_e m(i, e) \psi_S^{p-i}(V_{p+1}) V_{ep},$$

where

$$\psi_S^{p-i}(V_{p+1})V_{ep} = \begin{cases} V_{(p-i+e)p} - V_{(p-i-e)p} + (p-1)V_{ep} & \text{if } e < p-i, \\ V_{2ep} + (p-1)V_{ep} & \text{if } e = p-i, \\ V_{(p-i+e)p} + V_{(e+i-p)p} + (p-1)V_{ep} & \text{if } e > p-i. \end{cases} \quad (4.9)$$

It should be noted that  $V_{(p-i+e)p}$  in Eq. (4.9) cannot be  $V_{p^2}$  because  $e$  ranges from 1 to  $i-1$  for each  $1 \leq i \leq p-1$ . In what follows, for each  $r$  with  $1 \leq r \leq p-1$ , we compute the multiplicity of  $V_{rp}$  in  $\sum_i \psi_S^{p-i}(V_{p+1})[S^i(V_{p+1})]_{\text{Ind}}$ , which is equal to

$$\begin{aligned} & \sum_i \sum_{e < p-i} m(i, e)(V_{(p-i+e)p} - V_{(p-i-e)p} + (p-1)V_{ep}) \\ & + \sum_i m(i, p-i)(V_{2p(p-i)} + (p-1)V_{p(p-i)}) \\ & + \sum_i \sum_{e > p-i} m(i, e)(V_{(p-i+e)p} + V_{(e+i-p)p} + (p-1)V_{ep}). \end{aligned}$$

Thus, the coefficient of  $V_{rp}$  in  $\sum_i \psi_S^{p-i}(V_{p+1})[S^i(V_{p+1})]_{\text{Ind}}$  is given by

$$\begin{aligned} & \sum_{\substack{1 \leq i \leq p-1 \\ 1 \leq e \leq i-1 \\ i-e=p-r}} m(i, e) - \sum_{\substack{1 \leq i \leq p-1 \\ 1 \leq e \leq i-1 \\ i+e=p-r}} m(i, e) + \sum_{\substack{1 \leq i < p-r \\ r+1 \leq i}} m(i, r)(p-1) \\ & + m(p-r/2, r/2) + m(p-r, r)(p-1) \\ & + \sum_{\substack{1 \leq i \leq p-1 \\ p-i \leq e \leq i-1 \\ i-e=p-r}} m(i, e) + \sum_{\substack{1 \leq i \leq p-1 \\ 1 \leq e \leq i-1 \\ i+e=p+r}} m(i, e) + \sum_{\substack{p-r < i \leq p-1 \\ r+1 \leq i}} m(i, r)(p-1). \end{aligned}$$

Here, if  $r$  is odd,  $m(p-r/2, r/2)$  is set to zero. We now simplify the above summation.

First, note that if  $i+e=p-r$ , then  $1 \leq i \leq p-r-1$  since  $e=p-i-r \geq 1$ . In the same fashion, if  $i+e=p+r$ , then  $r+1 \leq i \leq p-1$  since  $e=p-i+r \leq p-1$ . As a consequence,

$$\begin{aligned} I &:= - \sum_{\substack{1 \leq i \leq p-1 \\ 1 \leq e \leq i-1 \\ i+e=p-r}} m(i, e) + \sum_{\substack{1 \leq i \leq p-1 \\ 1 \leq e \leq i-1 \\ i+e=p+r}} m(i, e) \\ &= - \sum_{1 \leq i \leq p-r-1} m(i, p-r-i) + \sum_{r+1 \leq i \leq p-1} m(i, p+r-i) \\ &= - \sum_{1 \leq i \leq p-r-1} m(i, p-r-i) + \sum_{1 \leq i \leq p-r-1} m(i+r, p-i) \quad (\text{by replacing } i \text{ by } i+r) \\ &= 0. \end{aligned}$$

The final equality follows from the identity  $m(i, p-r-i) = m(i+r, e)$ .

Second, note that

$$\begin{aligned}
 II &:= \sum_{\substack{1 \leq i < p-r \\ r+1 \leq i}} m(i, r)(p-1) + m(p-r, r)(p-1) + \sum_{\substack{p-r < i \leq p-1 \\ r+1 \leq i}} m(i, r)(p-1) \\
 &= (p-1) \sum_{r+1 \leq i \leq p-1} m(i, r) \\
 &= \begin{cases} \frac{p-1}{p} \left[ \binom{p}{p-2} + \binom{p+1}{p-2} + \cdots + \binom{2p-2-r}{p-2} \right] & \text{if } 1 \leq r \leq p-2, \\ 0 & \text{if } r = p-1. \end{cases}
 \end{aligned}$$

Third, note that if  $i - e = p - r$ , then  $m(i, e) = \binom{2p-1-r}{p-2} p^{-1}$ . Conversely, if

$$1 \leq i \leq p-1, \quad 1 \leq e \leq i-1, \quad i - e = p - r,$$

then  $1 \leq e = i - p + r < p - i$  and thus  $p - r + 1 \leq i < p - \frac{r}{2}$ . Consequently,

$$\sum_{\substack{1 \leq i \leq p-1 \\ 1 \leq e \leq i-1 \\ i-e=p-r}} m(i, e) = \begin{cases} \frac{r-1}{2} \binom{2p-1-r}{p-2} p^{-1} & \text{if } r \text{ is odd,} \\ \left(\frac{r}{2} - 1\right) \binom{2p-1-r}{p-2} p^{-1} & \text{if } r \text{ is even.} \end{cases}$$

Similarly,

$$\sum_{\substack{1 \leq i \leq p-1 \\ p-i \leq e \leq i-1 \\ i-e=p-r}} m(i, e) = \begin{cases} \frac{r-1}{2} \binom{2p-1-r}{p-2} p^{-1} & \text{if } r \text{ is odd,} \\ \left(\frac{r}{2} - 1\right) \binom{2p-1-r}{p-2} p^{-1} & \text{if } r \text{ is even.} \end{cases}$$

We also note that

$$m(p - r/2, r/2) = \begin{cases} 0 & \text{if } r \text{ is odd,} \\ \binom{2p-1-r}{p-2} p^{-1} & \text{if } r \text{ is even.} \end{cases}$$

Putting these together, it is evident that

$$III := \sum_{\substack{1 \leq i \leq p-1 \\ 1 \leq e \leq i-1 \\ i-e=p-r}} m(i, e) + m(p - r/2, r/2) + \sum_{\substack{1 \leq i \leq p-1 \\ p-i \leq e \leq i-1 \\ i-e=p-r}} m(i, e)$$

equals  $(r-1) \binom{2p-1-r}{p-2} p^{-1}$ .

Since the coefficient of  $V_{rp}$  in  $\sum_{i=1}^{p-1} \psi_S^{p-i}(V_{p+1})[S^i(V_{p+1})]_{\text{Ind}}$  equals  $I + II + III$ , we obtain

$$\sum_{i=1}^{p-1} \psi_S^{p-i}(V_{p+1})[S^i(V_{p+1})]_{\text{Ind}} = \sum_{r=1}^{p-1} x_r p^{-1} V_{rp}, \quad (4.10)$$

where

$$x_r = (r-1) \binom{2p-1-r}{p-2} + \sum_{i=1}^{p-r-1} (p-1) \binom{p-1+i}{p-2}.$$

Note that

$$\begin{aligned} & p \binom{2p-1-r}{p-2} - (r-1) \binom{2p-1-r}{p-2} - (p-1) \sum_{i=1}^{p-r-1} \binom{p-1+i}{p-2} \\ &= (p-r+1) \binom{2p-1-r}{p-2} - (p-1) \left[ -p + \binom{p+1}{p-1} + \binom{p+1}{p-2} + \cdots + \binom{2p-2-r}{p-2} \right] \\ &= (p-r+1) \binom{2p-1-r}{p-2} - (p-1) \binom{2p-1-r}{p-1} + p(p-1) \\ &= p(p-1). \end{aligned}$$

Here the second and third equalities come from the well-known formula  $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ . We now apply this identity to Eqs. (4.7) and (4.10) to obtain

$$p[S^p(V_{p+1})]_{\text{Ind}} - \sum_{i=1}^{p-1} \psi_S^{p-i}(V_{p+1})[S^i(V_{p+1})]_{\text{Ind}} = (p-2)V_p + \sum_{r=2}^{p-1} (p-1)V_{rp} + pV_{p^2}.$$

Conversely, by Eq. (4.3) and formulae (P4)–(P7), we derive the identity

$$\begin{aligned} & \psi_S^{p-i}(V_{p+1})V_{ip+1} \\ &= \begin{cases} (p-1)V_{ip} + V_{(i+1)p} - V_{p^2-1} + 2V_{p^2} - V_{(p-2i)p-1} - V_{ip+1} & \text{if } 1 \leq i \leq \frac{p-1}{2} \text{ odd,} \\ (p-2)V_{ip} - V_{p^2-1} + 2V_{p^2} - V_{(p-2i)p-1} + V_{(i+1)p-1} & \text{if } 1 \leq i \leq \frac{p-1}{2} \text{ even,} \\ (p-1)V_{ip} + V_{(i+1)p} - V_{p^2-1} + 2V_{p^2} + V_{(2i-p)p+1} - V_{ip+1} & \text{if } \frac{p+1}{2} \leq i \leq p-2 \text{ odd,} \\ (p-2)V_{ip} - V_{p^2-1} + 2V_{p^2} + V_{(2i-p)p+1} + V_{(i+1)p-1} & \text{if } \frac{p+1}{2} \leq i \leq p-2 \text{ even,} \\ (p-2)V_{ip} + 2V_{p^2} + V_{(2i-p)p+1} & \text{if } i = p-1. \end{cases} \end{aligned}$$

This implies that

$$\begin{aligned} \sum_{i=1}^{p-1} \psi_S^{p-i}(V_{p+1})V_{ip+1} &= \sum_{i=1}^{p-1} (p-1)V_{ip} - (p-2)V_{p^2-1} + 2(p-1)V_p^2 \\ &\quad - \sum_{i=1}^{\frac{p-1}{2}} V_{(p-2i)p-1} + \sum_{\substack{1 \leq i \leq \frac{p-1}{2} \\ \text{even}}} V_{(i+1)p-1} + \sum_{\substack{\frac{p+1}{2} \leq i \leq p-2 \\ \text{even}}} V_{(i+1)p-1} \quad (4.11) \end{aligned}$$

$$\begin{aligned} &+ \sum_{i=\frac{p+1}{2}}^{p-1} V_{(2i-p)p+1} - \sum_{\substack{1 \leq i \leq \frac{p-1}{2} \\ \text{odd}}} V_{ip+1} - \sum_{\substack{\frac{p+1}{2} \leq i \leq p-2 \\ \text{odd}}} V_{ip+1}. \quad (4.12) \end{aligned}$$

Since  $\{p - 2i: 1 \leq i \leq \frac{p-1}{2}\}$  is the same as  $\{i: 1 \leq i \leq p - 2, \text{ odd}\}$ , (4.11) equals

$$-\sum_{\substack{i=1 \\ \text{odd}}}^{p-2} V_{ip-1} + \sum_{\substack{i=2 \\ \text{even}}}^{p-2} V_{(i+1)p-1} = -\sum_{\substack{i=1 \\ \text{odd}}}^{p-2} V_{ip-1} + \sum_{\substack{i=2 \\ \text{odd}}}^{p-2} V_{ip-1} = -V_{p-1}.$$

In the same manner, we can show that (4.12) = 0 and hence

$$\sum_{i=1}^{p-1} \psi_S^{p-i}(V_{p+1})V_{ip+1} = \sum_{i=1}^{p-1} (p-1)V_{ip} - (p-2)V_{p^2-1} + 2(p-1)V_p^2 - V_{p-1}.$$

As a consequence,

$$\begin{aligned} \psi_S^p(V_{p+1}) &= pV_2 + p[S^p(V_{p+1})]_{\text{Ind}} - \sum_{i=1}^{p-1} \psi_S^{p-i}(V_{p+1})V_{ip+1} - \sum_{i=1}^{p-1} \psi_S^{p-i}(V_{p+1})[S^i(V_{p+1})]_{\text{Ind}} \\ &= pV_2 + (p-2)V_p + \sum_{r=2}^{p-1} (p-1)V_{rp} + pV_{p^2} \\ &\quad - \left( \sum_{r=1}^{p-1} (p-1)V_{rp} - (p-2)V_{p^2-1} + 2(p-1)V_p^2 - V_{p-1} \right) \\ &= pV_2 + V_{p-1} - V_p + (p-2)V_{p^2-1} - (p-2)V_{p^2}, \end{aligned}$$

as required.  $\square$

**Theorem 4.4.** If  $p = 2$ , then  $\psi_S^p(V_{p+1}) = 2V_2 - V_1$ .

**Proof.** The desired result is straightforward from Proposition 2.4(b), Theorem 3.5 and Eq. (2.1).  $\square$

Recall that we have already shown that if  $I_{p^2}$  is closed for the exterior powers, then

$$\psi_S^p(V_{p+1})(V_{p+1} - V_{p-1}) \equiv (p+1)(V_{p+1} - V_{p-1}) \pmod{I_{p^2}}.$$

If  $p$  is odd, then Theorem 4.3 states

$$\psi_S^p(V_{p+1}) \equiv pV_2 - (p-1)V_1 \pmod{I_{p^2}},$$

and hence

$$\begin{aligned} &\psi_S^p(V_{p+1})(V_{p+1} - V_{p-1}) - (p+1)(V_{p+1} - V_{p-1}) \\ &\equiv p(V_2(V_{p+1} - V_{p-1}) - 2(V_{p+1} - V_{p-1})) \pmod{I_{p^2}} \\ &\equiv p(V_{p+2} - V_{p-2} - 2V_{p+1} + 2V_{p-1}) \pmod{I_{p^2}} \\ &\equiv p(V_{2p} - V_{2p-2} - 2V_{p+1} + 3V_p - V_{p-2} - 2V_1) \pmod{I_{p^2}}. \end{aligned} \tag{4.13}$$



However, (4.13) cannot be zero modulo  $I_{p^2}$  because all the indecomposable modules in the last term are contained in  $\{V_{ap+j}: 0 \leq a \leq \frac{p-1}{2}, j = 1, 3, \dots, p\}$ , which is a  $\mathbb{Z}$ -basis of  $RS(p^2)/I_{p^2}$  (Proposition 2.6).

**Corollary 4.5.** *For every odd prime  $p$ , Conjecture 2.5 is not true.*

**Proof.** The proof follows from the natural embedding from  $RS(q) \rightarrow RS(pq)$  mapping  $V_n$  to  $V_n$  for all  $n \leq q$ .  $\square$

## References

- [1] G. Almkvist, R.M. Fossum, Decomposition of exterior and symmetric powers of indecomposable  $\mathbb{Z}/p\mathbb{Z}$ -modules in characteristic  $p$  and relations to invariants, *Lecture Notes in Math.* 641 (1978) 1–111.
- [2] D.J. Benson, *Representations and Cohomology I*, Cambridge University Press, Cambridge, 1995.
- [3] R.M. Bryant, Modular Lie representations of groups of prime order, *Math. Z.* 246 (2004) 603–617.
- [4] R.M. Bryant, M. Johnson, Adams operations on the Green ring of a cyclic group of prime-power order, *J. Algebra* 323 (2010) 2818–2833.
- [5] R.M. Bryant, M. Johnson, Periodicity of Adams operations on the Green ring of a finite group, *J. Pure Appl. Algebra* 215 (2011) 989–1002.
- [6] C.W. Curtis, I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, AMS Chelsea Publishing, Providence, RI, 2006, reprint of the 1962 original.
- [7] R.M. Fossum, *Decompositions revisited*, *Lecture Notes in Math.* 924 (1982) 261–295.
- [8] I. Hughes, G. Kemper, Symmetric powers of modular representations, Hilbert series and degree bounds, *Comm. Algebra* 28 (2000) 2059–2088.
- [9] F.M. Kouwenhoven, The  $\lambda$ -structure of the Green rings of cyclic  $p$ -groups, *Proc. Sympos. Pure Math.* 47 (1987) 451–466.
- [10] S.-Y. Nam, Y.-T. Oh,  $\lambda$ -ring structure of the Green ring of a cyclic  $p$ -group, *J. Algebra* 338 (2011) 92–113.
- [11] R.J. Shank, D.L. Wehlau, Decomposing symmetric powers of certain modular representations of cyclic groups: symmetry and spaces, *Progr. Math.* 278 (2010) 169–196.